Wave–Current Interactions in Finite Depth

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ABSTRACT

The energy, momentum, and mass-flux exchanges between surface waves and underlying Eulerian mean flows are considered, and terms in addition to the classical wave "radiation stress" are identified. The formulation is made in terms of the vertically integrated flow. The various terms are identified with other analyses and interpreted in terms of physical mechanisms, permitting reasonable estimates of the associated depth dependencies. One term is identified with the integrated "CL vortex force" implemented, for example, in simulations of Langmuir circulation. However, as illustrated with a simple example of steady refraction across a shear zone, other terms of the same order can significantly alter the results. The classic example of long waves forced by short-wave groups is also revisited. In this case, an apparent singularity arising in shallow water is countered by finite-amplitude dispersion corrections, these being formally of the same order as the forced long-wave response, and becoming significant or dominant as shallow water is approached.

1. Introduction

As waves are strained and refracted by currents, exchanges of mass, momentum, and energy occur between the waves and mean flow. Longuet-Higgins and Stewart (1962, 1964) described the net "excess flux of momentum due to the presence of waves," and, in analogy to optics, named it the "radiation stress" (noting a slight grammatical inconsistency but bowing to historical usage). Gradients in the radiation stress (momentum flux) of the waves are reflected in changes to the mean field, so the momentum of the combined system is conserved. Because energy and momentum are exchanged between the waves and mean flow, additional analysis is needed to determine the partitioning between the two. Earlier work (Longuet-Higgins and Stewart 1960, 1961) described the wave variations for a few cases, providing the basis to describe (e.g.) the generation of group-bound-forced long waves. Analyses of wave variations are facilitated by use of an adiabatic invariant, the "wave action" (Bretherton and Garrett 1968; Whitham 1974), defined as the intrinsic wave energy divided by the intrinsic frequency ("intrinsic" meaning evaluated in a frame moving with the mean flow). In the absence of the dissipation or generation of waves, the net flux of wave action is conserved.

The concept of surface wave radiation stress has proven useful in many scenarios, including wave-induced "set down" outside the surf zone and "setup" inside as waves shoal and break (Longuet-Higgins and Stewart 1964; Bowen et al. 1968); generation of longshore currents by obliquely incident waves (Bowen 1969; Longuet-Higgins 1970a,b); and the interaction of freely propagating long and short surface waves (Longuet-Higgins 1969; Hasselmann 1971; Garrett and Smith 1976; and many others). In particular, Hasselmann (1971) pointed out the following two neglected effects: 1) the "virtual mass source" resulting from convergences of the wave-induced mass flux that can lead to exchanges of potential energy, and 2) changes in wave momentum that absorb some of the radiation stress gradients. Garrett and Smith (1976) combined the radiation stress and mass balance into a consistent framework, resolving these issues. Using action conservation to account for variations in the wave momentum budget, and subtracting this from the radiation stress divergence, Garrett (1976) derived an effective "wave force" on the mean flow, and suggested a mechanism for the generation of Langmuir circulation, which is a prominent form of motion found in the wind-driven

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surface mixed layer (Langmuir 1938; Craik and Leibovich 1976, hereinafter CL76; Craik 1977; Leibovich 1980; Li et al. 1995; Skyllingstad and Denbo 1995; McWilliams et al. 1997; McWilliams and Sullivan 2000; Phillips 2002). A key term in this force arises from refraction of the waves by current shear, which had previously been neglected. As shown later (Leibovich 1980; Smith 1980), this "refraction force" is equivalent to a vertically integrated form of the "CL vortex force" derived by CL76 (also Craik 1977; Leibovich 1977).

This paper focuses on the physical interpretation of the terms in the interaction equations. The interpretation emphasizes separation into Eulerian mean, wave-mean, and interaction (mixed) terms. While much of the analysis is a reformulation of previous work, some new aspects here concern the 1) extension of the formulation of Garrett (1976) to include finite-depth effects, 2) recognition that finite-amplitude dispersion corrections are often of the same order as the effects of the currents on wave propagation, and 3) educated guesses that are made about the vertical structure associated with each forcing term, based on comparisons with other analyses and the resulting physical intuition. While mathematical rigor and explicit assessment of neglected terms are parallel goals, the emphasis on physical interpretation means that, for example, the equations are carried through in dimensional form, rather than forming explicit nondimensional small-parameter expansions. An attempt is made to make clear the assumptions employed, both explicit and implicit.

The effects of waves on the Eulerian mean flow are shown in two forms: as a correction to the use of radiation stress, and as an expression for a "wave force" like that shown for deep-water waves by Garrett (1976). Application is illustrated with two examples: 1) the refraction of waves by parallel shear, and 2) the classic problem of long waves forced by groups of shorter waves. In the refraction example, including only the wave refraction force (equivalent to the vertically integrated CL vortex force) without including both an additional mass acceleration term and the Eulerian mean flow response would lead to incorrect results. In the forced long-wave problem, nonlinear propagation terms are of the same order as those retained in the wave-forcing equation, and must be considered. In the shallow-water limit, these act to counter a singularity that would otherwise arise. While pressure fields measured in finite depth have been found consistent with such second-order theory (Herbers and Guza 1991; Herbers et al. 1994), horizontal velocities measured in the laboratory (e.g., Groeneweg and Klopman 1998; also Kemp and Simons 1982, 1983; Swan 1990; Jiang and Street 1991; S. G. Monismith et al. 1996, unpublished manuscript, hereinafter M96) and in the field (Smith 2006) show persistent differences from the simple theory, indicating a need for further work and understanding, particularly regarding the vertical structure of the response (which is not addressed rigorously here).

2. Momentum equations for waves and currents

For many purposes the dynamics of a fluid with a free surface can be simplified by integrating in depth and time averaging over the higher-frequency waves and turbulence. Care is required in averaging over the surface waves to properly account for the exchanges of mass, momentum, and energy with the mean flow. It is also important to identify and evaluate the assumptions employed.

Here the vertically integrated momentum budget is examined, including both mean flow and waves. Turbulence is neglected. For simplicity, the waves are considered as locally monochromatic. This "carrier wave" is allowed to vary spatially and temporally, consistent with (and interacting with) variations in the underlying medium; in general, these variations will be assumed have a larger scale in both time and space, as in Bretherton and Garrett (1968). The budget is divided into Eulerian mean, wave, and mixed quantities. The wave momentum budget is evaluated to second order in wave quantities, using dispersion, wavenumber evolution, and conservation of wave action (Bretherton and Garrett 1968; Whitham 1974). Use is also made of the "radiation stress" as defined and evaluated by Longuet-Higgins and Stewart (1962, 1964). By subtracting the waves' momentum budget from the total, the net effect of the waves on the mean momentum budget is deduced, extending the results of Garrett (1976) to finite-depth water. The exposition parallels Smith (1990), but with emphasis on the underlying physics.

a. Vertically integrated total momentum equations

In the following, the vertical coordinate z is treated separately from the horizontal ones x, y to facilitate vertical integration. Here, z is positive upward. The corresponding velocities are w and $u_{x,y}$.The indices i and j are allowed to run through the values x, y for the two horizontal components of a variable. The summation convention is used: repeated indices i or j are summed over the two horizontal components. Partial differentiation with respect to t is denoted as ∂_t, and with respect to x, y as either ∂_i, ∂_j or ∂_x, ∂_y.

For simplicity, we shall neglect viscosity, rotation, compressibility, and stratification. For inviscid flow in a nonrotating frame of reference, the horizontal momentum equation is

$$\partial_t(\rho u_i) + u_j\partial_j(\rho u_i) + w\partial_z(\rho u_i) + \partial_i(\hat{p} + \rho g z) = 0, \quad (2.1)$$

where $\hat{p}$ is pressure. The first three terms are the material derivative—the acceleration of a moving material parcel of water results from gravity and the pressure gradient. Conservation of mass combined with incompressibility yields continuity in the form of

$$\partial_i u_i + \partial_z w = 0. \quad (2.2)$$

Both (2.1) and (2.2) apply for $-h < z < \zeta$, where the water is assumed to extend from a stationary but horizontally (slowly) varying depth $-h(x, y)$ to a fluctuating surface at $z = \zeta$.

With uniform density and incompressibility, ρ may be set to 1 without loss of generality, and can be dropped from the equations. Defining the kinematic pressure $p = \hat{p}/\rho + gz$ and using (2.2), (2.1) can be rewritten in the form

$$\partial_t u_i + \partial_j(u_i u_j) + \partial_z(u_i w) + \partial_i p = 0. \quad (2.3)$$

This form expresses the local change of momentum (velocity) in terms of the gradients of the fluxes of the i component of momentum (fluxes in all three directions) and the pressure. This will facilitate comparisons with the radiation stress formulation after wave averaging.

The kinematic boundary conditions at the free surface ζ and bottom $-h$ are

$$\partial_t\zeta + u_i\partial_i\zeta - w = 0 \quad \text{at} \quad z = \zeta \quad \text{and} \quad (2.4)$$

$$u_i\partial_i h + w = 0 \quad \text{at} \quad z = -h. \quad (2.5)$$

In general, $-h$ may be defined as any material surface below which the wave motion is negligible. For example, in deep water, one may conceptualize a "wave layer" between the surface and $-h$ that is thin compared to other motions of interest, as in the long-wave and short-wave problem (Hasselmann 1971; Garrett and Smith 1976; Smith 1986, 1990). Near shore, placing $-h$ at the water–sediment boundary is a natural choice; then, it is reasonable to assume $\partial_t h = 0$ (a notable exception would be the generation of tsunamis by underwater slides).

Vertical integration of (2.3), combined with boundary conditions (2.4) and (2.5), results in

$$\partial_t\left(\int_{-h}^{\zeta} u_i\,dz\right) + \partial_j\left(\int_{-h}^{\zeta} u_i u_j\,dz\right) + \partial_i\left(\int_{-h}^{\zeta} p\,dz\right) = (p\partial_i\zeta)_{z=\zeta} + (p\partial_i h)_{z=-h}. \quad (2.6)$$

The terms on the right arise from commuting the vertical integral with the spatial derivative and applying the boundary conditions. This form is adopted to conform to the definition of the waves' radiation stress (Longuet-Higgins and Stewart 1962, 1964), discussed below.

Next, let the flow be separated into mean and wave components $u_i = \overline{u}_i + u_i'$, where an averaging operator $\overline{(\;)}$ is defined to remove the waves' oscillatory motions, for example, $\overline{u_i'} = 0$. The first term of (2.6) becomes

$$\partial_t\left(\overline{\int_{-h}^{\zeta} u_i\,dz}\right) = \partial_t\left(\int_{-h}^{\overline{\zeta}} \overline{u}_i\,dz\right) + \partial_t\left(\overline{\int_{\overline{\zeta}}^{\zeta} u_i\,dz}\right) \equiv \partial_t M_i^m + \partial_t M_i^W, \quad (2.7)$$

where

$$M_i^m \equiv \int_{-h}^{\overline{\zeta}} \overline{u}_i\,dz \quad (2.8)$$

is the mean current momentum, and, Taylor expanding from the mean surface,

$$M_i^W \equiv \overline{\int_{\overline{\zeta}}^{\zeta} u_i\,dz} \approx \left(\overline{\zeta' u_i'} + \frac{1}{2}\overline{\zeta'^2}\partial_z\overline{u}_i + \frac{1}{2}\overline{\zeta'^2\partial_z u'_i} + \cdots\right)_{z=\overline{\zeta}} \quad (2.9)$$

is the net wave momentum. In practice, this is normally truncated to the first term, because the mean vertical shear is assumed small, and the third term is of third order in wave quantities. The presence of nonnegligible vertical shear introduces several effects, including modification of the dispersion relation (Stewart and Joy 1974; Valenzuela 1976), and requires definition of an "effective mean advection velocity of the waves" U_i^a (Smith 1990). Here, the mean vertical shear terms in (2.9) (and below) will be neglected for simplicity (but it is noted that this is an area of active research). Formally, this amounts to assuming that $\partial_z\overline{u}_i \ll \sigma$, where σ is the (radian) wave frequency.

The second term of (2.6) yields

$$\overline{\int_{-h}^{\zeta} u_i u_j\,dz} = \int_{-h}^{\overline{\zeta}} \overline{u}_i\overline{u}_j\,dz + \int_{-h}^{\overline{\zeta}} \overline{u_i' u_j'}\,dz + \overline{\int_{\overline{\zeta}}^{\zeta} u_i u_j\,dz}. \quad (2.10)$$

The first term on the right is the Eulerian mean flow momentum flux, the second is part of the mean wave-induced momentum flux (or radiation stress), and the third term, which was neglected by Hasselmann (1971), may be evaluated by Taylor expansion and rendered into the form

$$\overline{\int_{\bar{\zeta}}^{\zeta} u_i u_j \, dz} \approx \left[\bar{u}_i \left(\overline{\zeta' u_j'} + \frac{1}{2} \overline{\zeta'^2} \partial_z \bar{u}_j \right) + \bar{u}_j \left(\overline{\zeta' u_i'} + \frac{1}{2} \overline{\zeta'^2} \partial_z \bar{u}_i \right) + \cdots \right]_{z=\bar{\zeta}} = U_i M_j^W + M_i^W U_j, \tag{2.11}$$

where U_i is the mean velocity at the mean surface (Garrett 1976). Note that a triple product of wave quantities has apparently been neglected, which would add the term $\overline{\zeta' u'^2}$. This would be more appropriately included below in (2.16) (the definition of wave radiation stress).

The mean pressure is separated into a part p^m, which would exist without the waves and a wave part $\overline{p^w}$. As defined, $\overline{p^w}$ is not zero but approximately $-\rho \overline{w'^2}$, as noted by Longuet-Higgins and Stewart (1962, 1964). Here an alternative to their explanation is provided. The instantaneous pressure at the instantaneous surface must equal the atmospheric pressure: $p(\zeta) = p^a$. To evaluate the average pressure at the mean level $\bar{\zeta}$, the pressure is Taylor expanded about the mean level using the instantaneous vertical pressure gradient $\partial_z p = -(g + \partial_t w)$. This yields

$$\overline{p(\bar{\zeta})} = \overline{p^a} - \overline{\zeta'(g - \partial_t w)} \approx \overline{p^a} + \partial_t(\overline{\zeta' w'}) - \overline{w' \partial_t \zeta'} = \overline{p^a} - \overline{w'^2}. \tag{2.12}$$

The first term in this result fits the definition of p^m, so the second term is $\overline{p^w}|_{z=\bar{\zeta}}$ (to second order in wave quantities). It is straightforward to show that this result extends though depth as well. Taking into account vertical acceleration, the pressure on a material surface displaced by a wave remains constant with respect to wave phase, taking the value that it would have in the absence of the waves; the above argument then applies directly. As an aside, this implies that for measurements using pressure as a proxy for depth, such as from a conductivity–temperature–depth (CTD) probe, the wave deflections are removed to at least second order in the wave slope. This mean wave pressure is in a sense an artifice arising from averaging in an Eulerian frame of reference: the water parcel whose average position is at depth z experiences just the appropriate "mean pressure" $p^m(z)$ at every instant at the displaced depth $(z + \zeta)$. Thus, for example, gradients of $-\rho \overline{w'^2}$ are not felt by any actual fluid parcel, but they do appear in the Eulerian mean equations (here they appear as part of the radiation stress). In any case, with these definitions, the pressure term [third term of (2.6)] becomes

$$\overline{\int_{-h}^{\zeta} p \, dz} = \int_{-h}^{\bar{\zeta}} p^m \, dz + \int_{-h}^{\bar{\zeta}} \overline{p^w} \, dz + \overline{\int_{\bar{\zeta}}^{\zeta} p \, dz}, \tag{2.13}$$

in which the first term on the right is the unperturbed mean pressure, the second term is part of the radiation stress, and the last term is the potential energy of the waves $(g\overline{\zeta'^2}/2)$, or (to lowest order) one-half of their total energy.

Neglecting vertical shear, the total momentum equation can be written in the form

$$\partial_t M_i^m + \partial_j \left[\int_{-h}^{\bar{\zeta}} (\bar{u}_i \bar{u}_j + \delta_{ij} p^m) \, dz \right] + \partial_t M_i^W + \partial_j (S_{ij} + U_i M_j^W + M_i^W U_j) = k_i G^W + [p^m(\bar{\zeta}) - \overline{w'^2}|_{\bar{\zeta}}] \partial_i \bar{\zeta} + p^m(-h) \partial_i h, \tag{2.14}$$

where

$$k_i G^W = \overline{(p^a \partial_i \zeta)_{z=\zeta}} \tag{2.15}$$

is the assumed to provide the input to wave momentum from the wind, and

$$S_{ij} \equiv \overline{\int_{-h}^{\zeta} (u_i' u_j' + \delta_{ij} p^w) \, dz} \tag{2.16}$$

is the radiation stress as defined by Longuet-Higgins and Stewart (1962, 1964). Note that Longuet-Higgins and Stewart (1962, 1964) evaluate S_{ij} to second order in wave quantities, assuming a locally flat bottom and negligible vertical shear; using their result imposes these limits as well. In particular, note that this implies neglect of the triple-product term mentioned above, $\overline{\zeta' u'^2}$. In practice, wave skewness can be large, and this term may well be significant compared to $U_i M_j^W$ and $M_i^W U_j$. This is suggested as a topic for further research.

Equation (2.14) separates the momentum budget into mean momentum evolution (the first two terms) plus a variety of wave, mixed wave/current, and pressure forcing terms. The wind input to surface waves via pressure–slope correlation was recast as $k_i G$. A corresponding bottom term would arise with form drag over a rough bottom, but it is neglected here. The mean wave pressure becomes negligible at $z = -h$, in accordance with the approximation $\overline{p^w} = -\overline{w'^2} \to 0$. The mean wave pressure times the mean surface slope term $(-\overline{w'^2}\partial_i\bar{\zeta})$ was noted by Hasselmann (1971) and assumed negligible; because mean surface slopes are in general extremely small, it is neglected hereafter as well. The mean pressure (p^m) times mean slope terms can either be neglected or recombined on the left-hand side to move the mean pressure gradient back inside the integral, consistent with common practice; the above arrangement facilitates use of S_{ij} as evaluated by Longuet-Higgins and Stewart (1962, 1964). Were viscous stresses included, surface and bottom shear stresses would appear on the right-hand side. For the vertically integrated flow these can be added in an ad hoc fashion, using (e.g.) a bulk drag formulation.

In many circumstances, wave terms other than the radiation stress gradient $\partial_j S_{ij}$ can be neglected, and to do so has become standard practice. It is therefore useful to recast (2.14) in terms of forcing the mean momentum by the radiation stress gradients plus a variety of "extra" terms,

$$\partial_t M_i^m + \partial_j\left(\int_{-h}^{\bar{\zeta}} \bar{u}_i\bar{u}_j\,dz\right) + \int_{-h}^{\bar{\zeta}} \partial_i p^m\,dz = -\partial_j S_{ij} - \partial_j(M_i^W U_j) - \partial_j(U_i M_j^W) + (k_i G^W - \partial_t M_i^W). \qquad (2.17)$$

The first term on the right is the radiation stress gradient. The next term can be considered as a correction to the net flux of wave momentum, modifying advection by the group velocity concealed in S_{ij} [see (2.21) below] to include advection by the mean flow U_j also. The third term embodies both straining and refraction interactions between the waves and the current. The final pair of terms represents any net imbalance between the local input and the local rate of change of the wave momentum. Implicit also is the assumption that wave dissipation (e.g., breaking) results in a transfer of wave momentum to the mean flow. One deficiency in this formulation is that some terms in (2.17) conceal mixtures of physical effects, and hence vertical structures, and also mixtures of adiabatic and diabatic effects.

An alternative is to consider the evolution of the total momentum $M_i^T \equiv M_i^m + M_i^W$. To provide a direct comparison with Phillips (1977), it is useful to 1) assume the mean velocity is uniform over the depth $H \equiv \bar{\zeta} + h$ so that the second term in (2.17) integrates to HU_iU_j; and 2) define a total transport velocity $U_i^T = U_i + M_i^W/H$. Then, (2.14) can be rearranged in the form

$$\partial_t M_i^T + \partial_j(U_i^T M_j^T) + \partial_j(S_{ij} - M_i^W M_j^W/H) + \partial_i\left(\int_{-h}^{\bar{\zeta}} p^m\,dz\right) = k_i G^W + p^m(\bar{\zeta})\partial_i\bar{\zeta} + p^m(-h)\partial_i h. \qquad (2.18)$$

It is now noted that 1) Phillip's definition of the "excess momentum flux" differs from that of Longuet-Higgins and Stewart, and is given by the two terms inside the brackets of the third term; 2) he assumes the nonwave-induced part of the mean pressure p^m is hydrostatic; and 3) he neglects the terms on the right-hand side; this is the same result (see Phillips 1977, his section 3.6). It can also be seen that once S_{ij} is evaluated (below), the difference in definitions of the radiation stress term is small. This formulation shares the same deficiencies as those of (2.17): the terms conceal mixtures of different physical effects. Also, wave effects have to be explicitly accounted for in comparisons with observations, because these are generally Eulerian in nature. On the other hand, it is often simpler to evaluate, essentially deferring most complexities to the evaluation of the waves.

Waves also affect the surface boundary condition (Hasselmann 1971; Garrett and Smith 1976). The surface kinematic condition (2.4), Taylor expanded in ζ' about $\bar{\zeta}$ and averaged, leads to (to second order in wave quantities)

$$\partial_t\bar{\zeta} + \bar{u}_j\partial_j\bar{\zeta} - \bar{w} = -\partial_j M_j^W \quad \text{at} \quad z = \bar{\zeta}, \qquad (2.19a)$$

or in vector form,

$$\partial_t\bar{\zeta} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\zeta} - \bar{w} = -\nabla\cdot\mathbf{M}^W. \qquad (2.19b)$$

In an Eulerian framework, variations in wave-induced mass flux act as sources and sinks of fluid at the mean surface $\bar{\zeta}$. For $\bar{\zeta} \neq 0$, as when the large-scale flow is also a wave, this implies a transfer of potential energy. This mass-flux condition at the height of the mean surface was the essential point raised by Hasselmann (1971) in his refutation of the "maser mechanism" (Longuet-Higgins 1969) for long-wave growth. This might also be important near shore where setup can be significant.

Invoking nondivergence in the interior, integrating over depth, and including the surface and bottom boundary conditions yields conservation of mass over the water column,

$$\partial_t \overline{\zeta} + \nabla \cdot \mathbf{M}^m = -\nabla \cdot \mathbf{M}^W. \tag{2.20}$$

Near a straight shoreline and in steady state, for example, this mass balance implies that the alongshore-averaged shore-normal Eulerian transport is equal and opposite to the average shore-normal wave-induced mass flux (or Stokes' transport).

To evaluate the net interaction, expressions for the wave momentum and associated radiation stress terms are required. For most purposes, it is sufficient to specify the wave energy, momentum, and radiation stresses to second order, and for this it is usually sufficient to specify the waves to first order (Whitham 1974). This is taken up in the next two sections (but see also section 4d).

b. Linear dispersion and radiation stress

The linear dispersion equation for gravity waves in finite-depth water and with negligible currents relates the intrinsic radian frequency σ and wavenumber magnitude k,

$$\sigma^2 = gk \tanh(kH) \tag{2.21}$$

(Phillips 1977), where g is the local gravity, and $H \equiv h + \overline{\zeta}$ is the total mean depth. This leads to the intrinsic phase speed $c \equiv \sigma/k$, and group velocity $c_i^g \equiv \partial_{k_i}\sigma$ (where k_i are the components of k). The net propagation of the waves with currents is $c_i^g + U_i$, and the apparent frequency is Doppler shifted, as discussed below.

The net excess flux of momentum resulting from the waves, or radiation stress S_{ij}, was evaluated by Longuet-Higgins and Stewart (1962, 1964) for negligible vertical mean shear and locally uniform depth. The result can be written

$$S_{ij} = M_i^W c_j^g + HJ\delta_{ij} = E^W(k_i c_j^g/\sigma) + HJ\delta_{ij}, \tag{2.22}$$

where $E^W = g\overline{\zeta'^2}$ is the wave energy (recall $\rho \equiv 1$), and (with the summation convention)

$$HJ \equiv \frac{1}{2} H(\overline{u'^2} - \overline{w'^2})$$
$$= M_j^W\left(c_j^g - \frac{1}{2}c_j\right) = E^W\left(kc^g/\sigma - \frac{1}{2}\right) \tag{2.23}$$

is a finite-depth term that vanishes in deep water (the reason for including H in the definition will be seen shortly). This "J term" has the same form as the irrotational wave-induced stress term derived by Rivero and Arcilla (1995). The J term acts dynamically like a pressure term.

The tensor form $M_i^W c_j^g$ for the radiation stress was employed by Garrett (1976) for deep-water gravity waves; inclusion of the J term generalizes the form to finite depth (Smith 1990). This form is also valid including surface tension, with appropriate evaluation of M_W, c, c^g, and E_W, as can be verified by detailed comparison with section 3 of Longuet-Higgins and Stewart (1964). However, this estimate of the radiation stress does not include a sloping mean surface or bottom. There are concerns regarding usage of these results in the context of nonnegligible slopes, but these are beyond the scope of this paper. It also neglects a term related to wave skewness (as mentioned above).

Using these identities and definitions, (2.14) can be rewritten in the form

$$\partial_t M_i^m + \partial_j\left(\int_{-h}^{\overline{\zeta}} \overline{u}_i\overline{u}_j\,dz\right) + \int_{-h}^{\overline{\zeta}} \partial_i p^m\,dz + \partial_t M_i^W + \partial_j[M_i^W(c_j^g + U_j)] = k_i G^W - \partial_j(M_j^W U_i) - \partial_i(HJ). \tag{2.24}$$

Equation (2.24) separates the total momentum budget into Eulerian mean flow momentum evolution (top row of terms) and wave momentum evolution along a ray (middle row) versus wave growth and interaction terms (bottom row).

c. Wave momentum evolution

To close the equations, another expression for the wave momentum evolution is needed. This is found from conservation of wave action and wave crests. The treatment considers a group of waves as it propagates and varies in height, direction, and wavelength. Consistent with the approximations used to derive action conservation, the waves in the group are assumed to act locally like a plane wave, with the variations occurring over time and space scales that are large compared to those of the waves. This is fundamentally different from a spectral density formulation; for example, as a packet is compressed and becomes shorter, the action density increases within the packet, but the resulting spectral density from the packet is spread over a wider range of wavenumbers.

Wave momentum can be written $M_i^W = Ak_i$, where $A = E^W/\sigma$ is the "wave action," k_i is the wavenumber, and E^W is the wave energy. Wave action is an adiabatic invariant (Bretherton and Garrett 1968; Whitham 1974), so the action evolution equation for a wave packet is straightforward to write as a ray equation,

$$\partial_t A + \partial_j[A(c_j^g + U_j)] = G^W - D^W, \tag{2.25}$$

where G^W and D^W represent the wave growth and dissipation, respectively, in terms of action. The growth is attributed to wind. For example, a Miles-style growth term would have the form $G^W = \beta A$, where β depends on the wind (Miles 1957, 1960). Similarly, the action dissipation term D^W represents both wave breaking and viscous decay, and the corresponding wave momentum is transferred to the mean flow. Bottom friction, where momentum is lost from the combined mean flow + wave system, would introduce another term; but this is neglected here. Note that D^W must have stronger wave amplitude dependence than G^W for the waves to have a stable equilibrium amplitude.

The propagation velocity of wave action includes both the wave group velocity and the underlying mean flow $c_i^g + U_i$. In practice, typical mean flows (i.e., aside from exceptional cases such as the alongshore flow near shore, tidal jets from inlets, and, perhaps, western boundary currents) are often of order $(ak)^2 c^g$, which is the same size as finite-amplitude corrections to c^g. If the effect of the flow on the waves is important to the net interaction, finite-amplitude dispersion should probably also be considered (e.g., see section 4d).

To obtain an equation for wave momentum $M_i^W = k_i A$, (2.25) is combined with an equation for the wavenumber k_i. Conservation of wave crests yields

$$\partial_t k_i + \partial_i(\sigma + k_j U_j) = 0 \tag{2.26a}$$

or

$$\partial_t k_i + (c_j^g + U_j)\partial_j k_i = -k_j \partial_i U_j - \partial_H \sigma \partial_i H \tag{2.26b}$$

(Phillips 1977). Variations in the medium other than depth, such as apparent gravity or surface tension, are neglected. Note that for steady currents (and nonmoving bottom) the apparent frequency $\omega = \sigma + k_j U_j$ is constant over the whole domain, simplifying the evaluation of the wavenumber field in that case. Combining (2.26b) with (2.25) yields a wave momentum evolution equation

$$\partial_t M_i^W + \partial_j[M_i^W(c_j^g + U_j)] = k_i\{\partial_t A + \partial_j[A(c_j^g + U_j)]\} + A[\partial_t k_i + (c_j^g + U_j)\partial_j k_i] = k_i(G^W - D^W) - M_j^W \partial_i U_j - J\partial_i H. \tag{2.27}$$

The last term arises from $\partial_H \sigma$ combined with the definitions of E^W, A, and J; this term accounts for adiabatic effects of shoaling on wave momentum. The second-to-last term accounts for the adiabatic variations of M_i^W resulting from current gradients (both straining and refraction). For steady flows ($\partial_t k_i = 0$), all of these effects are implicitly included in (2.17) and (2.18) via the radiation stress gradient.

d. Wave forcing of mean flows

Combining (2.14), (2.22), and (2.27) leads to an equation describing the net effect of the waves on the Eulerian mean flow:

$$\partial_t M_i^m + \partial_j\left(\int_{-h}^{\bar{\zeta}} \bar{u}_i \bar{u}_j \, dz\right) + \int_{-h}^{\bar{\zeta}} \partial_i p^m \, dz = F_i^W, \tag{2.28}$$

where the mean pressure gradient is moved back inside the integral, and the wave force F_i^W acting on the mean flow can be written as

$$F_i^W \equiv k_i D^W + M_j^W(\partial_i U_j - \partial_j U_i) - U_i \partial_j M_j^W - H\partial_i J, \tag{2.29a}$$

or

$$\mathbf{F}^W \equiv \mathbf{k} D^W + \mathbf{M}^W \times (\nabla \times \mathbf{U}) - \mathbf{U}(\nabla \cdot \mathbf{M}^W) - H\nabla J. \tag{2.29b}$$

The first term represents the dissipation of wave momentum (e.g., via breaking); the momentum is assumed to be transferred directly to the mean flow. The second term is the reaction to wave refraction: as the waves are refracted, their momentum changes; an equal and opposite change must occur in the mean flow to conserve the total. The third term represents the momentum required account for the mass source/sink at the surface: this is introduced or removed with a momentum corresponding to the mass times the mean flow speed at the surface. The final term accounts for forcing by the "J term" gradient less that accounted for by the wave evolution [see (2.27)]. This form for the wave force acting on the mean flow was identified by Garrett (1976) for deep-water gravity waves. Modification for finite depth simply adds the J term (Smith 1990).

In a typical application, a map of the wave momentum is generated for an instantaneous configuration of the depth, currents, and incident waves, and the results

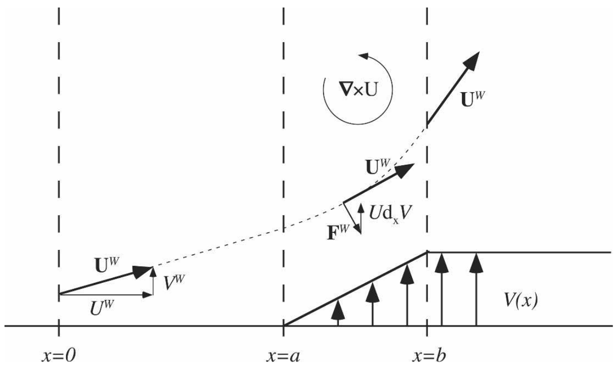

FIG. 1. Example 1: steady waves incident across a parallel shear zone. The waves are refracted by the shear. The component of wave force parallel to the shear (y or vertical as shown here) is counteracted by a mean flux to the left ($-x$) of y momentum [(3.3), (3.4); arrow denoted Ud_xV]. The component perpendicular (x) can only be balanced by a pressure gradient, resulting in setup.

are substituted into (2.29) or (2.17) [or (2.18)] to evaluate the wave field and the consequent slower evolution of the mean flow. Which of the wave–current interaction equations is simpler [(2.17), (2.18), or (2.28) and (2.29)] depends on the problem addressed.

3. Steady shear

Steady parallel horizontal shear provides a simple yet illustrative case (Fig. 1). For simplicity, wave input and dissipation are negligible. Currents and statistical properties of the waves are steady and uniform in y (i.e., the waves can be incident at an angle θ^W, but all mean flow and mean wave quantities are steady and uniform in y, so both ∂_t, $\partial_y \equiv 0$). Currents $U(x)$, $V(x)$ are assumed uniform in the vertical.

First, it is useful to define depth-mean wave transport velocities [note the relation to the wave-related part of the total transport velocity defined for (2.18)]

$$U^W \equiv M_x^W/\rho H \quad \text{and}$$
$$V^W \equiv M_y^W/\rho H. \tag{3.1}$$

Then, continuity (2.20) can be written as

$$\partial_x[H(U + U^W)] = 0. \tag{3.2}$$

The equation for y momentum [from (2.28) and (2.29)] becomes

$$\partial_x(HUV) = -HU^W\partial_x V - V\partial_x(HU^W). \tag{3.3}$$

The first term on the right is the one associated with refraction, the second with the spinup of the mass flux lost from the waves to velocity V. The term on the left can be separated into the two parts, $HU\partial_x V + V\partial_x(HU)$. Then, combining the second of these with the last term on the right, and applying continuity (3.2), the above reduces to

$$HU\partial_x V = -HU^W\partial_x V. \tag{3.4}$$

If the shear and depth are nonzero anywhere, this constrains the mean transport to counter the wave transport $U = -U^W$ at that point. Then, because of continuity, they must cancel everywhere. There is no constraint on $V(x)$ beyond it being steady parallel shear flow. Any shear profile is consistent with the assumptions, and, in the absence of friction and forcing, simply continues unmodified. This illustrates the following two points. 1) The "spinup" to the mean velocity V of mass injected by a decreasing wave mass flux (say) is balanced by the "spin down" of mass removed by the corresponding increase in mean mass flux imposed by continuity. 2) The wave refraction force in the y direction is apparently countered by the mean advection of y momentum across its gradient [as indicated by (3.4)]. Another way to view this result is that for the shear to remain at a fixed location in x, and therefore satisfy the assumption of steady state, the net Lagrangian transport in the x direction ($U + U^W$) must be zero.

The x momentum budget is not quite as simple. With the same assumptions, the mean flow x momentum equation is [again from (2.28) and (2.29)]

$$\partial_x(HU^2) + gH\partial_x\bar{\zeta} = HV^W\partial_x V - U\partial_x(HU^W) - H\partial_x J. \tag{3.5}$$

Similar to above, the first term is split into $HU\partial_x U + U\partial_x(HU)$; the latter part combines with the second term on the right, and is eliminated because of continuity. All remaining occurrences of H are outside of the derivatives, so this can be divided. The result is

$$g\partial_x\bar{\zeta} = V^W\partial_x V - \partial_x J - \frac{1}{2}\partial_x U^2. \tag{3.6}$$

In terms of the depth-mean wave transport velocities, we can rewrite (2.23) in the form

$$J = |\mathbf{U}^W|\left(c^g - \frac{1}{2}c\right) = (U^{W2} + V^{W2})^{1/2}\left(c^g - \frac{1}{2}c\right). \tag{3.7}$$

From the y momentum solution [(3.4) and discussion after], $U = -U^W$, so $U^2 = U^{W2}$, and either can be used in evaluating (3.6). Alternatively, comparing this to the terms in J, it appears reasonable to neglect this term. It is consistent to neglect the U^2 term but not the $V^W\partial_x V$ one, because there is no similar constraint on the size of the shear $\partial_x V$; also, neglecting the shear term would amount to neglecting the whole problem. In contrast to the y momentum budget, however, the wave refraction force in the x direction is not countered by mean advection, and must instead be balanced by a surface slope. Even for deep-water waves (where the J term is zero), shear can induce setup via the refraction force (the first term on the right). For waves incident from a low angle (i.e., nearly parallel to the shear), this refraction force can dominate.

In both the x and y momentum balances, the wave force term resulting from spinup of the mass-flux divergence was countered by the corresponding spin down of the mean mass-flux divergence, which (for a steady state) must exactly balance because of continuity. Some models currently incorporate the CL vortex force only [implicitly including the first term on the right in (3.3)], but neglect the other (spinup) term. This could yield an incomplete cancellation and an erroneous result; however, this would only happen if the mass-flux divergence was included in calculating the mean flow, but excluded from the wave force.

To determine the setup, the response of the waves is needed. However, it is not necessary to show explicit solutions: the main points of this example are clear already from the form of the equations. 1) The net interaction does not modify the shear profile $V(x)$, because the wave force component parallel to the shear (y direction) is countered by the mean advection of momentum [see (3.3) and (3.4), and the discussion thereafter]. This is consistent with there being no change in the x flux of the wave's y momentum as well [as can be seen from (2.27)], and with the system remaining in steady state. 2) Refraction of the waves by the shear does alter the surface setup across the shear zone, because of the component of the wave force perpendicular to the shear [see (3.7)]; this can be assisted or countered by shoaling effects [the J term in (3.7)]. 3) The term arising from spinup of the wave's mass-flux divergence [the last term in (3.3)] is balanced by the spin down of the corresponding mean flow mass-flux divergence enforced by continuity. 4) The results are consistent with the general principle that the effects of (irrotational) waves can neither create nor destroy vorticity (though it can rearrange it).

4. Longer waves forced by wave groups

The second example is the Eulerian motion induced by groups of waves or swell propagating into still water. The analysis here resembles "method 2" of Longuet-Higgins and Stewart (1962, hereinafter LHS62). The following simplifications are made: wave input and dissipation are negligible; the induced motion is irrotational; the induced response U/c^g is small, of order $(ak)^2$, as is M^W/hc^g, so all terms in F^W involving products of M^W and U can be neglected [order $(ak)^4$] compared to the virtual pressure term $H\nabla J$ [order $(ak)^2$] and the surface mass-source term [also order $(ak)^2$]; the bottom is locally flat, $h = constant$, and $\overline{\zeta}/h$ is very small, so h can replace H; and the x axis is aligned with M^W, as is the response, so subscripts x or y are unnecessary (e.g., $M^W = M_x^W$).

a. Shallow-water-forced waves

The simplest case is one where the forced wave is long compared to the depth but the free waves are not. Then, $\overline{u} \equiv U$ is approximately uniform from $\overline{\zeta}$ to $-h$. To lowest order, the momentum balance [(2.28) and (2.29)] becomes

$$\partial_t U + \partial_x(g\overline{\zeta}) = -\partial_x J, \tag{4.1}$$

where the mean pressure is hydrostatic (consistent with the long-wave assumption).

A solution is sought propagating with the constant wave group velocity c^g, so $\partial/\partial t$ can be replaced by $-c^g\partial/\partial x$. This results in

$$c^g U - g\overline{\zeta} = J, \tag{4.2}$$

where the constant of integration is chosen so that $U = \overline{\zeta} = 0$ when there are no waves [this is just Eq. (3.20) of LHS62]. The lowest-order mass conservation Eq. (2.20) becomes

$$hU - c^g\overline{\zeta} = -M^W, \tag{4.3}$$

where the constant of integration is chosen as above. The solution is

$$U = -\frac{gM^W + c^g J}{gh - (c^g)^2}, \tag{4.4}$$

and

$$\overline{\zeta} = -\frac{c^g M^W + hJ}{gh - (c^g)^2}, \tag{4.5}$$

consistent with the results in LHS62. Note that when the primary (free) waves are deep-water waves ($J = 0$), the forcing results entirely from the virtual mass source at the surface $\nabla \cdot \mathbf{M^W}$. As the depth increases further, so $(c^g)^2 \ll gh$, the forced flow balances the Stokes' transport $hU = -M^W$, so there is zero net (integrated) Lagrangian transport. Conversely, as the group velocity approaches the shallow-water limit the solution blows up, violating the initial assumption of a small response. Clearly, exact cancellation of the terms in the denominator is sensitive to small corrections to the group velocity due to finite-depth and finite-amplitude waves. These limits are pursued further in sections 4c and 4d.

In terms of the depth-mean wave transport velocity $M^W = hU^W$ and $J = U^W(c^g - c/2)$, and the above takes the form

$$U = -U^W\left[\frac{gh + (c^g)^2 - \frac{1}{2}c^g c}{gh - (c^g)^2}\right], \tag{4.6}$$

and

$$\bar{\zeta} = -hU^W\left[\frac{2c^g - \frac{1}{2}c}{gh - (c^g)^2}\right]. \tag{4.7}$$

b. Finite-depth-forced waves

Again following LHS62, the results can be generalized to deeper water (or shorter groups) with the additional assumption that the vertical structure of the forced response in $\bar{u}$ and p^m is the same (as is the case for a surface wave–like potential flow response). Define θ such that

$$\int_{-h}^{\bar{\zeta}} \bar{u}\,dz \equiv \theta(h + \bar{\zeta})U \approx \theta hU, \tag{4.8}$$

where U is the mean horizontal surface velocity. For example, if the wave groups force a simple harmonic long wave of wavenumber K, the resulting potential flow solution would yield

$$\theta \approx \frac{\tanh(Kh)}{Kh}. \tag{4.9}$$

By assumption, the pressure has the same form,

$$\int_{-h}^{\bar{\zeta}} p^m\,dz \approx \theta hg\bar{\zeta}. \tag{4.10}$$

To lowest order in the mean quantities, the momentum equation becomes

$$\partial_t(\theta hU) + \partial_x(\theta hg\bar{\zeta}) = -h\partial_x J. \tag{4.11}$$

Using $\partial_t \to -c^g\partial_x$, integrating with c^g and h assumed independent of x, and dividing by θh, this becomes

$$c^g U - g\bar{\zeta} = J/\theta. \tag{4.12}$$

Again, integration constants are chosen so there is no motion or elevation in the absence of waves. Mass conservation yields

$$\partial_t\bar{\zeta} + \partial_x(\theta hU) \approx -\partial_x M^W, \tag{4.13}$$

or

$$c^g\bar{\zeta} - \theta hU = M^W. \tag{4.14}$$

Combining these,

$$U = -\frac{gM^W + c^g J/\theta}{\theta gh - (c^g)^2} = \frac{M^W}{\theta h}\left[\frac{\theta gh + (c_0^g)^2 - \frac{1}{2}c_0^g c_0}{\theta gh - (c^g)^2}\right], \tag{4.15}$$

and

$$\bar{\zeta} = -\frac{c^g M^W + hJ}{\theta gh - (c^g)^2} = M^W\left[\frac{2c^g - c/2}{\theta gh - (c^g)^2}\right]. \tag{4.16}$$

The results have the roughly same form as in (4.4) through (4.7), but with h replaced by θh (except in the J term). As noted in LHS62, this can be extended to arbitrarily shaped groups by Fourier expanding the forcing terms M^W and J, and (to lowest order in $\bar{u}$ and $\bar{\zeta}$) summing the results.

c. Deep-water-forced waves

In deep-water the wave force becomes negligible, and the response is driven entirely by the mass-flux condition at the surface boundary,

$$\partial_t\bar{\zeta} - W = -\partial_x M^W, \tag{4.17}$$

where W is the vertical velocity associated with the response (forced long wave). For a wave group with envelope wavenumber K propagating with group speed c_g as contemplated here, ∂_t is replaced by $-c_g\partial_x$. Assuming the response is irrotational and nondivergent, it should have a depth dependence of the form e^{Kz}. By continuity, W can be replaced by $\partial_x(U/K)$, where U is the surface value of the horizontal response current. The momentum equation and surface boundary condition then reduce to

$$\partial_x(-c^g U + g\bar{\zeta}) = 0, \tag{4.18}$$

and

$$\partial_x(U/K - c^g\bar{\zeta} + M^W) = 0, \tag{4.19}$$

respectively. Choosing constants of integration so that $U = 0$ when $\bar{\zeta} = 0$, the partial derivative operators can simply be dropped. The solution takes the form

$$\bar{\zeta} = (c^g/g)U \quad \text{and} \tag{4.20}$$

$$U = -\frac{gM^W}{g/K - (c^g)^2} \tag{4.21}$$

[as in (4.15), (4.16) with $J = 0$ and $h\theta = K^{-1}$]. For deep-water waves, $(c^g)^2 = (½c)^2 = ¼(g/k)$. Using also $M^W = (2k)^{-1}U^S$, where $U^S = a^2k^S\sigma$ is the surface Stokes drift associated with the waves (and is not to be confused with U^W, see also discussion in section 5), (4.21) can be written as

$$U = -U^S\left[\frac{(K/2k)}{1-(K/4k)}\right]. \quad (4.22)$$

A wave group must contain at least one full wave, so $K < k$. Thus, the surface current response should be less than $(2/3)U^S$ in magnitude. In contrast, both laboratory measurements (e.g., Klopman 1994, as referenced in Groeneweg and Klopman 1998; also Kemp and Simons 1982, 1983; Swan 1990; Jiang and Street 1991; M96) and some recent open-ocean measurements (Smith 2006) indicate a surface response equal in magnitude to the surface Stokes drift, which is a somewhat stronger response. It might be speculated that some vertical structure is imposed on the response that differs from that assumed above. Because the above form seems to be implied for an irrotational, nondivergent Eulerian mean response, however, this is puzzling. This clearly warrants further study.

d. Finite-amplitude, finite-depth free waves

In the shallow-water limit, the first-order propagation speed terms in the denominators of (4.4) and (4.5) cancel, so higher-order terms must be considered. Among the next-order terms to consider are both finite-amplitude modifications of the phase and group velocities of the free waves and the back interaction of the forced velocity field on both the short-wave propagation and amplitude evolution. Proper treatment of finite-amplitude dispersion requires consideration of the actual spectrum of waves (Herbers et al. 2002). However, for strongly peaked spectra, and to provide a simple example, it is sufficient to consider a "carrier wave" with a slowly modulated amplitude, which is treated as if it were locally a unimodal plane wave, consistent with the approximation used in deriving action conservation (Bretherton and Garrett 1968).

For a unimodal wave train, the phase speed c and group speed c_g expand to second order in wave slope as

$$c = \left(\frac{g}{k}\tanh kH\right)^{1/2}\left[1 + \frac{(ak)^2}{16}(9 - 10\coth^2 kH + 9\coth^4 kH)\right] \quad \text{and} \quad (4.23)$$

$$c^g = \frac{1}{2}\left(\frac{g}{k}\tanh kH\right)^{1/2}\left\{1 + R + \frac{(ak)^2}{16}[9(5+R) - 10(5-3R)\coth^2 kH + 9(5-7R)\coth^4 kH]\right\}, \quad (4.24)$$

where

$$R \equiv \frac{2kH}{\sinh 2kH}. \quad (4.25)$$

These are derived from the nonlinear dispersion relation of Whitham (1974), which in turn is derived from an expansion in wave steepness ak, and is applied locally as if the wave were approximately a uniform plane wave. This expansion diverges in very shallow water: as kH decreases, the terms proportional to $(ak)^2\coth^4 kH$ eventually become large. To assess where this becomes a problem it is worth looking at some actual values. As an example, for 11-s waves incident with deep-water rms height $W = 1.5$ m, the term $(ak)^2\coth^4 kH$ approaches unity near 5-m water depth, while depth-limited breaking does not start until closer to 3-m depth (see Fig. 2). The effects of the finite-depth and finite-amplitude corrections on the wave phase and group speeds are illustrated in Fig. 3 for the same example wave. The finite-amplitude corrections are small except close to shore, as expected; the deep-water steepness is $ak \approx 0.025$, so the deep-water phase speed is increased by a factor of only 1.0003. The expansions appear to diverge near the inner limit at 5-m depth. In very shallow water, the terms involving $(ak)^2\coth^4 kH$ increase the phase speed but decrease the group speed with increasing amplitude; that is, the dispersive characteristics of the waves are enhanced, as seen most clearly in the ratio c^g/c^p (Fig. 4). This may help to retard wave front steepening as the waves shoal, and hence help to prolong the existence of a steep just-before-breaking waveform.

The above expansion is for frequency as a function of a given wavenumber ($\sigma = ck$); the speeds are written as functions of k, H, and amplitude a ($a = W/2$), not frequency σ. In practice, it is more common to specify the frequency (e.g., as in section 3); in this case dispersion has to be solved iteratively to determine k. To complicate the issue, in a typical application the wave amplitude itself is a function of the group velocity: the energy flux $E^W c^g$ is specified, so E^W (and hence the amplitude and steepness) depends inversely on c^g. Thus σ, h, and $E^W c^g$ are input parameters, and k, c, c^g, and E^W are found by iteration. To further complicate the issue, the mean response U is of the same order as the nonlinear terms of c^g, and so advection by U must be included as well. In principle the energy flux $E^W(c^g + U)$ is speci-

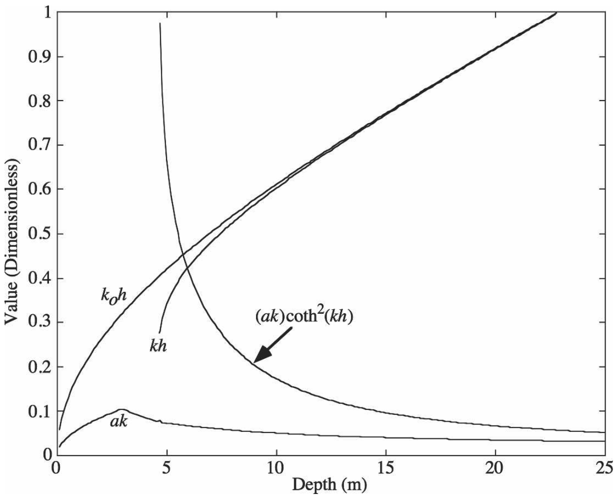

FIG. 2. Dimensionless parameters for surface waves in finite depth with finite amplitudes [(4.23)–(4.25)], except for "k_0h," which uses the linear k_0. The example calculations are for 11-s-period incident waves of 1.5-m rms height in deep water. Depth-limited breaking is shown for ak at depths less than 3 m.

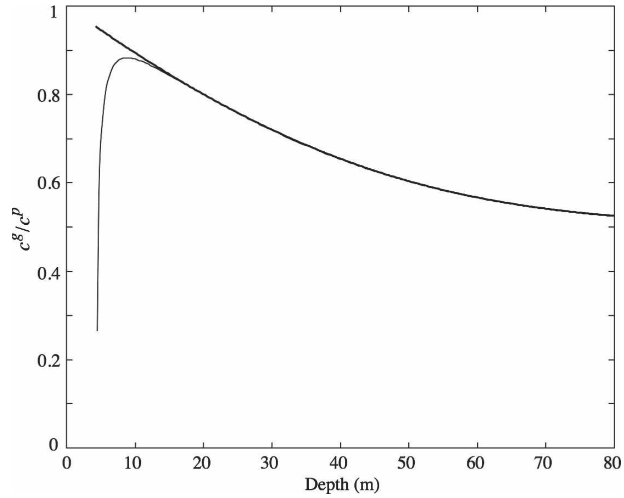

FIG. 4. Ratio of surface wave group speed over phase speed for an 11-s, 1.5-m incident wave: linear finite depth (thick line); finite-amplitude finite-depth solution (thin line). The finite-amplitude corrections become large very quickly as the water shoals.

fied at the outer boundary, and the system of coupled equations is solved as a set.

To obtain a simple solution for this group-forced long-wave problem, the modulations of the free-wave amplitude are assumed small relative to overall mean wave amplitude. Thus, for example,

$$M^W = M_0^W + \delta M_1^W + H.O., \tag{4.26}$$

where the subscript 0 denotes overall mean quantities and δ is a small parameter. The overall means must

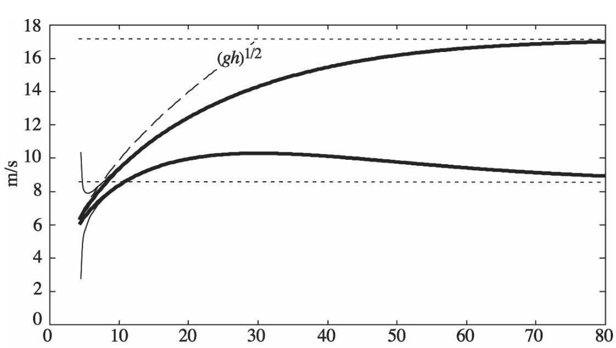

FIG. 3. Finite-depth and finite-amplitude surface wave phase and group velocities [Eqs. (4.23), (4.24)]. Linear limit (thick lines), finite amplitude (thin lines), linear shallow limit (long dashed lines), and linear deep-water limits (horizontal dashed lines). Phase velocities are the upper set of lines, group velocities are the lower set. For the example wave (11-s period, 1.5-m height), the finite-amplitude corrections are negligible in water deeper than about 10 m, and the finite/shallow approximation performs well to about 20 m, deeper than expected.

themselves satisfy the basic equations in (2.20), (2.27), and (2.28) with (2.29). The overall mean flow is assumed to be uniform in depth. For consistency, a steepness parameter $\varepsilon^2 \equiv (ak)^2$ is introduced. The transport velocities U_0 and M_0^W/H are of order $\varepsilon^2 c_0$, as are the nonlinear terms in the group velocity c_0^{gNL}. The modulations U_1/c_0, M_1^W/c_0, and $\overline{\zeta}_1/H$ are of the order of $\delta\varepsilon^2$. The modulation of group velocity $c_1^g = c_1^{gNL}$ is also of the order of $\delta\varepsilon^2 c_0$, but these will be seen to enter only in the wave evolution equation in (2.27).

Supposing that somewhere downstream there is a coast imposing no net x flux of mass, or (alternatively) picking a frame moving with the net mean Lagrangian flow, continuity (2.20) implies that

$$HU_0 + M_0^W = 0. \tag{4.27}$$

The total depth H should formally be H_0, but because $H_1 \equiv \overline{\zeta}_1$ this would be redundant. The mean wave momentum is most easily found from the conservation of action A_0 [(2.25)] and crests [(2.26a)], rather than from momentum [(2.27)]. First, a reference action flux B_∞ and (constant) encounter frequency σ_∞ are specified, nominally for deep-water and zero mean current. Then, action conservation can be written

$$A_0(c_0^g + U_0) = B_\infty. \tag{4.28}$$

The equations for frequency and dispersion combine to yield

$$\sigma_\infty = \sigma + k_0U_0 = k_0c_0 - k_0^2A_0/H, \tag{4.29}$$

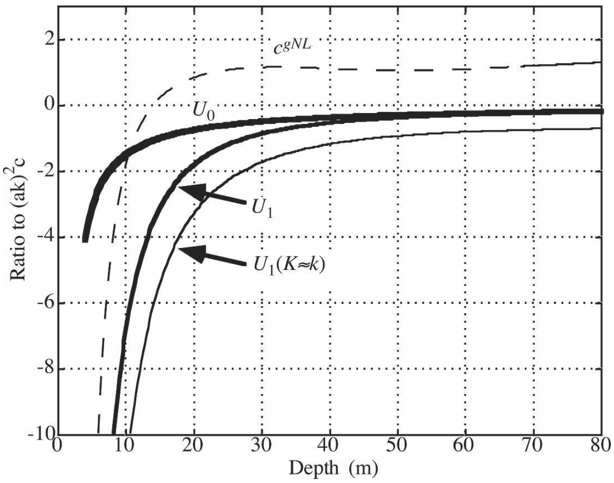

FIG. 5. Size of the nonlinear groups speed terms [dashed line; (4.24)], mean advection U_0 [thickest line; (4.28)], uniform-with-depth response U_1 [medium line; (4.37) with $\theta = 1$], and short-group limit response $U_1(K \approx k)$ [thin line; (4.37), with $\theta = \theta(k)$], all normalized by $(ak)^2c$.

where c_0 is the nonlinear phase speed for the overall mean conditions, and use was made of (4.37) and the relation $M_0^W = k_0A_0$. The nonlinear terms of c_0 are proportional to $\varepsilon^2 \equiv (ak)^2$ [see (4.23)]. The wave energy is $E^W = \sigma A = ½ga^2$, so the steepness parameter is evaluated as

$$\varepsilon^2 = A_0(2\sigma k_0^2/g). \tag{4.30}$$

As illustrated in Fig. 5, the nonlinear dispersion terms are nominally the same size as the Doppler shift, so these should be evaluated together in (4.29). Because the Doppler-shifted dispersion relation (4.29) is a complicated function of *k, h,* and ε^2, it is solved via iteration. The overall mean wave parameters are evaluated through iteration of (4.28) and (4.29). It is useful to note that at order ε^2, σ_∞ can be substituted for σ in (4.30) for use in (4.29), and that each additional pass through (4.28) and (4.29) provides a correction that is smaller by another order ε^2 relative to the last [i.e., only two passes are needed to get $O(\varepsilon^2)$ results].

To proceed and evaluate the effects of order δ modulations of the overall means, all such quantities (denoted with subscript 1) are assumed to be approximately steady in a frame of reference moving at the constant net group velocity $(c_0^g + U_0)$, so $\partial_t \rightarrow -(c_0^g + U_0)\partial_x$. Note in particular that the mean group velocity c_0^g includes a mean finite-amplitude correction (it is not just the linear group velocity). In addition, let the forced long wave have a characteristic wavenumber K, and (as in section 4b) assume the pressure and horizontal velocity have the same vertical structure. Thus, for example, the integral of the order δ horizontal velocity results in θU_1, where U_1 is the order δ velocity at the surface. Then, the order δ equation for continuity (2.20) becomes

$$\theta H U_1 - (c_0^g + U_0)\bar{\zeta}_1 = -M_1^W \tag{4.31}$$

and the momentum equations [(2.28) with (2.29)] yields

$$(c_0^g - U_0)\theta H U_1 - g\theta H\bar{\zeta}_1 = HJ_1 + U_0M_1^W, \tag{4.32}$$

where the constants of integration are chosen so the order δ quantities have zero overall means. The resulting solution is

$$\bar{\zeta}_1 = -\left[\frac{M_1^W c_0^g + J_1}{\theta g H - (c_0^g)^2 + U_0^2}\right] \quad \text{and} \tag{4.33}$$

$$\begin{aligned} U_1 &= -\left[\frac{gM_1^W + (c_0^g + U_0)(HJ_1 + U_0M_1^W)/\theta H}{\theta g H - (c_0^g)^2 + U_0^2}\right] \\ &= -\left[\frac{gM_1^W + c_0^g J_1/\theta}{\theta g H - (c_0^g)^2 + U_0^2}\right] - U_0\left(\frac{\bar{\zeta}_1}{\theta H}\right) \\ &\quad - \left[\frac{U_0^2 M_1^W/\theta H}{\theta g H - (c_0^g)^2 + U_0^2}\right]. \end{aligned} \tag{4.34}$$

Comparing these with (4.15) and (4.16), we see that in addition to including nonlinear terms in c_0^g, this analysis has added U_0^2 to the denominators, and a couple of other new terms in U_1. It is useful to assess the relative sizes of the various terms. Of particular interest is the behavior as kH becomes small. In this limit,

$$gH - (c_0^g)^2 \rightarrow gH\left[(kH)^2 + \frac{9}{8}\varepsilon^2(kH)^{-4} + \cdots\right]. \tag{4.35}$$

The first term comes from the finite-depth linear dispersion relation, while the second arises from the nonlinear correction to group velocity. The nonlinear dispersion terms diverge in the shallow-water limit, restricting the validity of the above to depths where the second term above is small relative to 1; however, this analysis extends estimation of the forced long-wave amplitudes into somewhat shallower depths than the linear solution. The third term in the denominators of (4.33) and (4.34) U_0^2 is of the order of ε^4, and so can be neglected relative to the nonlinear propagation term. The numerators of (4.33) and the first term of the second line of (4.34) are both of the order of $\delta\varepsilon^2$. The equivalent numerator in the second term of (4.34) is of the order of $\delta\varepsilon^4$, while the last is of the order of $\delta\varepsilon^6$; thus, these last two terms may also be neglected. The final assessment based on this ordering is that the mean advection terms (involving U_0) can be neglected, but the nonlinear dispersion terms can be important, particularly in (moderately) shallow water.

At the leading order, by similar arguments, $J_1 = (M_1^W/H)(c_0^g - ½c_0)$; so the above solution can also be written in the form

$$\bar{\zeta}_1 = -M_1^W\left[\frac{2c_0^g - \frac{1}{2}c_0}{\theta gH - (c_0^g)^2 + U_0^2}\right] \approx -M_1^W\left[\frac{2c_0^g - \frac{1}{2}c_0}{\theta gH - (c_0^g)^2}\right] \quad \text{and} \tag{4.36}$$

$$U_1 = -\frac{M_1^W}{\theta H}\left[\frac{\theta gH + (c_0^g + U_0)\left(c_0^g - \frac{1}{2}c_0 + U_0\right)}{\theta gH - (c_0^g)^2 + U_0^2}\right] \approx -\frac{M_1^W}{\theta H}\left[\frac{\theta gH + (c_0^g)^2 - \frac{1}{2}c_0^g c_0}{\theta gH - (c_0^g)^2}\right]. \tag{4.37}$$

These are precisely as in (4.15) and (4.16), but for the conceptual addition of the mean nonlinear correction terms to the group velocity. A similar overall result regarding the combined effect of nonlinear dispersion and advection by the long-wave field was obtained with a different approach by McWilliams et al. (2004; see their section 6.5).

The modified long-wave forcing is illustrated in Fig. 6, in terms of the ratio of the long-wave current U_1 to the driving wave mass-flux velocity $M_1^W/\theta H$. The singularity in the long-wave forcing is suppressed as finite-amplitude modifications of the group velocity (in particular) result in additional terms of the same order or larger than the remainder from previously retained terms (which nearly cancel out). It can also be seen that as the depth increases the ratio approaches unity (as mentioned above).

For gentle waves and swell (small ak), the finite-amplitude parameter $(ak)^2 \coth^4 kH$ can exceed 0.5 well before the waves break, at which point the corrections to wave phase and group velocities are significant, and probably unphysical. Additional terms or different expansion strategies are needed in such shallow water; however, this analysis extends the validity of the long-wave evaluation somewhat farther into shallow water than the linear solution. It applies to very weak bottom slope; in the presence of finite slopes, the response has been observed to lag behind the short-wave groups, and to evolve as the waves shoal (Janssen et al. 2003; Battjes et al. 2004).

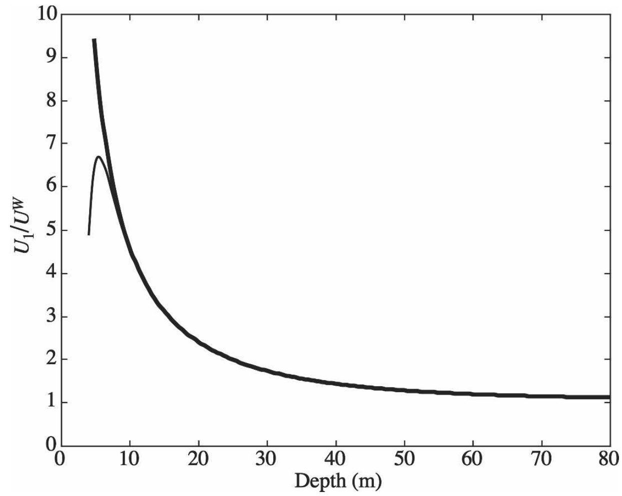

FIG. 6. The long-wave forcing ratio: amplitude of the Eulerian response U_1 to the amplitude of the Stokes' transport $U^W = M^W/h$: linear finite-depth solution (thick line); finite amplitude and finite depth (thin line). Note that the response is always larger than the wave group Stokes transport: the net transport of the combined system as a group passes is backward relative to the propagation direction. The depth and degree of nonlinearity are scaled as for 11-s waves with 1.5-m wave height in deep water.

5. Discussion: Physical interpretation and vertical structure

The formulation and derivation in this paper have so far been in terms of vertically integrated quantities. However, most of the terms in the wave force (2.29) can be identified with similar terms evaluated in other works (to be cited), and the resulting physical insight permits reasonable estimates of the vertical structure.

Reevaluation of the second and fourth terms of the wave force (2.29b) as functions of depth is facilitated by an alternative approach to the basic equations. For a homogeneous fluid as contemplated here, it is useful to return to (2.1) (with $\rho \equiv 1$) and employ the vector identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\nabla\times\mathbf{u}) \tag{5.1}$$

to obtain

$$\partial_t\mathbf{u} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + p\right) = \mathbf{u}\times\boldsymbol{\omega}, \tag{5.2}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity [Batchelor 1967; e.g., his (5.1.1) and (6.2.3)], and all three components are included in the vectors.

To simplify the analysis, the x axis is aligned with wave propagation, so all wave quantities are independent of y. Then, $\overline{\mathbf{u}\cdot\mathbf{u}} = \overline{\mathbf{u}}\cdot\overline{\mathbf{u}} + (\overline{u'^2} + \overline{w'^2})$. Next divide the pressure into the mean and wave part as before, and recall that the wave-induced part is $\overline{p^w} = -\overline{w'^2}$ throughout the fluid. Then, the wave-averaged gradient term in (5.2) can be written

$$\overline{\nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + p\right)} = \boldsymbol{\nabla}\left[\frac{1}{2}\overline{\mathbf{u}}\cdot\overline{\mathbf{u}} + p^m + \frac{1}{2}(\overline{u'^2} - \overline{w'^2})\right]. \tag{5.3}$$

Comparing the last term with (2.23), it is recognized as J from the radiation stress. This J term acts dynamically like a hydrostatic pressure gradient, and is also identified with an irrotational wave-induced Reynolds stress $\overline{u'w'}$ (Rivero and Arcilla 1995). The rest of these gradient terms are just the normal mean flow momentum terms.

To proceed, the wave-averaged result from the term on the right in (5.2) is needed. The equation for vorticity is obtained readily by taking the curl (which eliminates the gradient term),

$$\partial_t\boldsymbol{\omega} = \boldsymbol{\nabla}\times(\mathbf{u}\times\boldsymbol{\omega}). \tag{5.4}$$

As before, we separate the flow into a mean and wave components, now including the vorticity (so $\boldsymbol{\omega} = \overline{\boldsymbol{\omega}} + \boldsymbol{\omega}'$). The "wave components" of vorticity arise solely from perturbations of the mean vorticity because of straining and tilting by the wave motion. Consistent with the previous assumptions, the mean flow is smaller than the orbital velocities, and the mean flow gradients are also small; thus, these wave-induced vorticity fluctuations can be neglected in the wave evolution equations. However, as noted by CL76, these wave-correlated components give rise to effects that are not negligible in evaluating the evolution of the mean fields. Paralleling closely the arguments of CL76 [their (8)–(13)], the wave-induced vorticity fluctuations resulting from stretching and tilting the mean are evaluated and then used to evaluate the last term of (5.2). Because these fluctuations are induced by the wave motion, they can be assumed to propagate with the waves; that is, ∂_t can be replaced by $-c^{-1}\partial_x$. Noting also that the wave motions (and therefore also wave-induced fluctuations) are independent of y, the vertical fluctuating component of vorticity is evaluated [here the subscript on ω denotes the component, while the velocity components are (u, v, w) as before]: $\partial_t\omega_z' = \partial_x(w'\overline{\omega}_x - u'\overline{\omega}_z - \overline{u}\omega_z')$ or $(\overline{u} - c)\partial_x\omega_z' = \partial_x(w'\overline{\omega}_x - u'\overline{\omega}_z)$, or, neglecting $\overline{u}$ relative to c,

$$\omega_z' = \overline{\omega}_z(u'/c) - \overline{\omega}_x(w'/c), \tag{5.5}$$

Similarly, the x component is

$$\omega_x' = \overline{\omega}_x(u'/c) - \overline{\omega}_z(w'/c). \tag{5.6}$$

The y-component fluctuations of vorticity are zero. This is self-consistent because 1) there is no straining, and hence no vortex stretching, in that direction, and 2) the motions are approximately uniform in y and therefore cannot tilt or twist y vorticity into or out of the other components at the wave frequency. Using these results in (5.2) (in the term on the right-hand side) and averaging over the waves, the only nonzero wave-induced contribution is the y component

$$(\overline{\mathbf{u}'\times\boldsymbol{\omega}'})_y = \overline{w'\omega_x'} - \overline{u'\omega_z'} = \overline{\omega}_z c^{-1}(\overline{w'^2} + \overline{u'^2}). \tag{5.7}$$

To complete the comparison with CL76, this is put in terms of the depth-resolved Stokes drift, defined as the difference between the average horizontal velocity at a fixed point and the average horizontal velocity of a material fluid parcel whose mean position would have been at that point in the absence of the waves. The depth-resolved Stokes drift $u^S(z)$ is evaluated to second order in wave quantities in terms of the local displacements ζ and η combined with the local gradients of velocity. The displacements are

$$\zeta'(z) = \int_{t_0}^{t} w'(z)\,dt \quad \text{and}$$
$$\eta_i'(z) = \int_{t_0}^{t} u_i'(z)\,dt, \tag{5.8}$$

where t_0 is chosen so $\overline{\zeta'} = 0$ and $\overline{\eta_i'} = 0$ in the absence of waves. To lowest order, the Stokes drift is

$$\begin{aligned} u_i^S(z) &\approx \overline{\zeta'\partial_z u_i'} + \overline{\eta_j'\partial_j u_i'} = \overline{\zeta'\partial_z u_i'} - \overline{u_i'\partial_j\eta_j'} \\ &= \overline{\zeta'\partial_z u_i'} + \overline{u_i'\partial_z\zeta'} = \partial_z(\overline{\zeta' u_i'}). \end{aligned} \tag{5.9}$$

Here use is made of the knowledge that $\overline{\eta_j' u_i'} = 0$ for surface waves, and that incompressibility implies $\partial_j\eta_j' + \partial_z\zeta' = 0$. Using the linear finite-depth solution for a monochromatic wave,

$$u_i^S(z) = a^2\sigma k_i\frac{\cosh 2k(h+z)}{2\sinh^2 kh} = k_i\sigma^{-1}(\overline{u'^2} + \overline{w'^2}), \tag{5.10}$$

where the amplitude a is defined so that $\overline{\zeta^2} = \tfrac{1}{2}a^2$. Aligning $\mathbf{k}$ with the x axis, this precisely matches the factor multiplying $\overline{\omega}_z$ in (5.7), yielding $\overline{\omega}_z u^S$ for that term, consistent with CL76.

The last form shown in (5.9) for the depth-resolved Stokes drift reveals that its vertical integral is just M_i^W [(2.9), truncating to second order]:

$$M_i^W \approx (\overline{\zeta' u_i'})_{z=\overline{\zeta}} = \int_{-h}^{\overline{\zeta}} \partial_z(\overline{\zeta' u'})\,dz. \tag{5.11}$$

With the assumption that the mean flow, and therefore $\overline{\omega}_z$ also, does not vary significantly over the depth influenced by the waves, the term on the right of (5.7) [and hence the wave-induced part of (5.2) as well] integrates directly to $\mathbf{M}^W \times (\boldsymbol{\nabla} \times \mathbf{U})$ (where now $\mathbf{M}^W$ and $\mathbf{U}$ include horizontal components only), recovering this "refraction force" term of the wave force (2.29b).

For the steady shear problem (section 3), the vertical structure of the refraction force would induce a secondary circulation. The wave refraction force drives the uppermost fluid more strongly than below [with a profile like that of the Stokes drift; see (5.7) and (5.10)], while the setup (pressure gradient) forces a more nearly uniform return flow with an equal and opposite vertically integrated mass flux.

The remaining terms in the wave force are not easy to identify and evaluate rigorously in the depth-resolving equations; hence, the following is somewhat speculative. The first term in (2.29b), dissipation of wave momentum (mainly via breaking), presumably acts like a surface stress on the mean flow, as has been shown for viscous dissipation (Phillips 1977). Alternatively, wave breaking may initially induce rollers (Svendsen 1984), which then transfer the momentum via interfacial (surface) stress to the mean. The third term in (2.29b) is an adjustment accounting for the fact that mass flux lost from the waves ($\boldsymbol{\nabla} \cdot \mathbf{M}^W$) appears not with zero velocity, but at the mean flow speed $\mathbf{U}$. Following this logic, the results should depend on the cause of the mass-flux divergence. Part of the change in mass-flux is because of dissipation, and this part should appear as a correction to the "wave dissipation stress" at the surface, using the mean surface velocity. Part is because of adiabatic adjustment of the waves to varying depth, and should be uniformly distributed in depth like J (thus involving the depth-mean velocity). The part resulting from refraction by currents should be distributed in depth like that term [e.g., (5.7)], and so would involve a depth-weighted-average velocity in the vertical integral form.

6. Conclusions

The analysis and examples have brought to light the following several points of interest:

1) The "radiation stress" associated with surface waves acts on the combined (wave + Eulerian mean) flow system. Wave evolution equations provide a way to isolate the effects on the mean flow versus changes in the waves. The analysis here extends Garrett's (1976) work to finite depth.
2) Care is needed to consider all appropriate terms. One term, the "refraction force" [$\mathbf{M}^W \times (\boldsymbol{\nabla} \times \mathbf{U})$], is identified with a vertical integral of the "CL vortex force" derived by Craik and Leibovich (CL76; Craik 1977; Leibovich 1977, 1980), which has been implemented in recent models simulating (e.g.) Langmuir circulation (Leibovich and Tandon 1993; Skyllingstad and Denbo 1995; McWilliams et al. 1997; Phillips 2002). However, as shown in a simple shear-refraction example, other terms of the same order may significantly alter the results.
3) The Eulerian transport in response to passing wave groups is at least as large as the Stokes transport of the driving wave groups. Thus, the net fluid transport of the combined system is backward relative to the wave propagation direction. The backward transport increases with shoaling.
4) The Eulerian group-forced response and finite-amplitude corrections to phase and group speeds are formally of the same order, $(ak)^2c$. This has two very different implications:
   (a) Finite-amplitude dispersion counters an apparent singularity in the shallow-water limit for the response.
   (b) Advection by the forced response is of the same size as the finite-amplitude effects, and should be included in analyses of nonlinear wave group evolution.
5) Additional work is needed in considering the vertical structure of wave-forced flows. Recent laboratory measurements of vertical profiles of waves and currents indicate a close correspondence between the estimated Stokes drift profile and that of the opposing Eulerian response (e.g., Groeneweg and Klopman 1998; Kemp and Simons 1982, 1983; Swan 1990; Jiang and Street 1991; M96). The simple analysis and interpretation given here does not explain these profiles. While complex numerical simulations do appear to reproduce the laboratory results (Groeneweg and Klopman 1998), physical interpretation and explanation of this response remain elusive.

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